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Semi-inhomogeneous solutions of the Kac model of Boltzmannn equations

H Cornille†

Service de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette Cédex, France

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Abstract. We construct semi-inhomogeneous solutions of the Kac model, for which the spatial dependence is only present in the odd-velocity part of the distribution. At the macroscopic level, the local density, the local energy, and the associated current components are determined and discussed, while at the microscopic level there exist two different classes of distributions; as an illustration we give two solutions explicitly. For one class, the distributions relax towards a Maxwellian equilibrium solution and they correspond either to a contraction or an expansion, while for the other they go to zero when the time increases to infinity. The space variable appears linearly and, in order to maintain the positivity, stays inside a finite interval. Assuming that the distributions are zero outside such intervals, a physical interpretation can be obtained for the whole space axis, provided appropriate elastic walls, sinks and sources are introduced. These boundary conditions seem more natural for the class of distributions with Maxwellian relaxation.

1. Introduction

It seems that the discovery of exact solutions can help our comprehension of the Boltzmann equation.

In the past this was clear for the homogeneous Boltzmann formalism with Maxwell interaction. In that case (Bobylev 1976, Krook and Wu 1976), an exact even-velocity distribution is known (called BKW even mode). This BKW even solution exists also (Ernst 1981) for the simpler $1+1+1$ dimensional (velocity v, time t, space x) Kac (1956) model; further other exact solutions have recently been obtained (Cornille 1984a) for this model.

In the inhomogeneous Boltzmann formalism with Maxwell interaction, there exists (Nikolskii 1964) a transformation generating exact inhomogeneous solutions from homogeneous ones. Applying this method to the BKW even mode, Bobylev has deduced an associated inhomogeneous one which unfortunately goes to zero when the time goes to infinity. However, in the Maxwell interaction case and Nikolskii transform, Tenti and Hui (1979) have shown that other exact solutions can be generated, provided sources and sinks are introduced. Unfortunately all the inhomogeneous solutions built up with this method cannot relax towards a Maxwellian equilibrium solution and tend to zero when t increases. In fact Nikolskii showed that they correspond to expansions (contractions) in a three-dimensional space. On the contrary for the Kac model, where the Nikolskii transform cannot be applied we have found and exact inhomogeneous

† Chercheur CNRS.

solution in a one-dimensional space (Cornille 1984b), which relaxes towards a Maxwellian. Trying to understand more deeply the origin of that solution it turns out that there exists a family of distributions which share similar properties at both the macroscopic and the microscopic levels.

Here, for the Kac model, we define an intermediate formalism called semi-inhomogeneous, where we assume that the position x enters only in the odd velocity part f^- of the distribution function $f(v, t, x) = f^+(v, t) + f^-(v, t, x)$. Then the local density, the local energy and the associated current components satisfy a system of closed equations which depend on only three moments of the cross sections. Therefore these macroscopic equations can be solved with very few constraints on the microscopic interactions. At the macroscopic level we find two different classes of solutions for which we can at the microscopic level associate two classes of distributions $f(v, t, x)$, and the previously found solution (Cornille 1984b) belongs to one of them.

In § 2 we establish and study the macroscopic equations. The equation for the semi-inhomogeneous Kac model is: $\partial_t f + v\partial_x f^- = \mathrm{col}(f)$ and we first remark that the only possible stationary solutions are either $f = 0$ or a Maxwellian $f = \exp(-v^2 \mathrm{const})$. The local density $N_0^+ = \int f \, dv$, and the local energy $N_2^+ = \int f v^2 \, dv$ are spatially uniform, while the associated current components $J_0 = \int f v \, dv$, $J_2 = \int f v^3 \, dv$ are linear in the x-space variable. They satisfy a system of closed equations which are integrable. The first determined macroscopic quantity is the local density. There exist two different classes of asymptotic behaviours: either $N_0^+ \to 1^\mp + O(\exp(-\mathrm{const}\, t))$ when t tends to infinity or $N_0^+ \to O(t^{-1})$ corresponding to *expansion* (*contraction*) in one case and *expansion* in the other. For these two different classes of local densities, the stationary solutions are either Maxwellian equilibrium solutions or zero, also with two associated different classes of other macroscopic quantities N_2^+, J_0, J_2. For instance, for the average flow velocity J_0/N_0^+ we find for large t either $xO(\exp(-\mathrm{const}\, t))$ or xt^{-1} showing that it becomes negligible in one case and not in the second. For the second class with inverse power-time behaviour, we notice the analogy with the Nikolskii transformed solutions obtained for the Boltzmann equation with Maxwell interaction (Tenti and Hui 1979). It is also worthwhile noticing that the local energy N_2^+ associated with a Maxwellian relaxation satisfied a second-order differential equation so that there exists the possibility of two different relaxation times.

In § 3, at the microscopic level, we construct a class of solutions $f(v, t, x)$ with inverse power time dependence for N_0^+, N_2^+, which go to zero when t goes to infinity. For simplicity we assume that the v, t dependences are degenerate as a product of v-dependent functions by t-dependent ones. These distributions satisfy a specular reflection property at an arbitrary $x = x_0$ value and there exists an interval $x \in x_0 - a$, $x_0 + a$, where they are positive for all t and v values.

In § 4, at the microscopic level, we consider the class of solutions associated with Maxwellian equilibrium distributions and for simplicity focus our attention on the previously determined one (Cornille 1984b). We sketch briefly its properties. Depending on the scattering model, we have either *contraction* or *expansion*. A distinction also occurs depending on whether we have one or two relaxation times. There exists a specular reflection boundary at $x = x_0$ and we can still define a class of distributions f with positivity preserved for x inside an interval $(x_0 - a, x_0 + a)$.

The great defect of the distributions determined in §§ 3 and 4 is that they necessarily violate positivity for x outside the above interval $(x_0 - a, x_0 + a)$. These intervals can be chosen arbitrarily large but must remain finite and our f solutions violate positivity when $|x| \to \infty$. In § 5 we define new distributions $\tilde{f}$ which are equal to f for x inside

(x_0-a, x_0+a) and zero outside these intervals. For these new distributions $\tilde{f}$ we have $\partial_t\tilde{f}+v\partial_x\tilde{f}=\operatorname{col}(\tilde{f})+v(S^+ + S^-)$ and the Kac equation has two supplementary terms that we must interpret. Looking at the first supplementary term, we find that it corresponds to *elastic walls*. Its amplitudes remain fixed for the solutions of § 4 relaxing towards a Maxwellian, and decrease to zero like t^{-1} for those of § 3. For the second term and Maxwellian equilibrium solution, it corresponds either to a *sink* (expansion case) or to a *source* (contraction case) and becomes negligible, going to zero exponentially in time. On the contrary for the solutions of § 3 (having $f=0$ for stationary solutions), this term decreases only like t^{-2} and remains important.

2. Equations and solutions for the macroscopic quantities

The inhomogeneous Kac model with $f^+=f(v,t)$ is

$$\partial_t f^+(v)+v\partial_x f^-(v)=\nu\int_{-\infty}^{+\infty}\mathrm{d}w\int_{-\pi}^{+\pi}\mathrm{d}\theta\,\sigma(\theta)(f^+(v')f^+(w')-f^+(v)f^+(w)) \qquad (1a)$$

$$\partial_t f^-(v)=\nu\int_{-\infty}^{+\infty}\mathrm{d}w\int_{-\pi}^{+\pi}\mathrm{d}\theta\,\sigma(\theta)(f^-(v')f^+(w')-f^-(v)f^+(w)) \qquad (1b)$$

where $f^{\mp}(v)$ means $f^+(v,t)$, $f^-(v,t,x)$, $\sigma(\theta)=\sigma(-\theta)$ is the cross section, $v'=v\cos\theta-w\sin\theta$, $w'=v\sin\theta+w\cos\theta$, and ν is the collision constant.

To start with, putting $\partial_t f^{\pm}=0$, we look at the possibile stationary solutions. There exists the trivial $f^+\equiv f^-\equiv 0$ solution and to search for non-trivial ones, we assume that f^+, $v^{-1}f^-$ belong to the L_2 spaces spanned by the Laguerre polynomials $L_n^{\mp 1/2}(v^2/2)$ or $f^+(2\pi)^{1/2}\exp(v^2/2)=F^+$, $f^-(2\pi)^{1/2}=(x_0-x)2^{-1/2}vF^-$ with

$$F^{\pm}=\sum L_n^{(\mp 1/2)}(v^2/2)(-1)^n D_n^{\pm}. \qquad (2)$$

Substituting into equations (1a) and (1b) we find $D_n^-=0\ \forall n$ or $f^-\equiv 0$ while the vanishing of the collision kernel on the LHS of (1a) leads to the only solution $f^+=\exp(-\text{const}\ v^2)$. So the stationary solutions are either the trivial $f\equiv 0$ or the Maxwellian solutions, we do not consider $f\equiv f^+\equiv \text{const}\neq 0$ which leads to an infinite local density.

In this part of the paper the discussion arises at the macroscopic level and concerns hydrodynamical quantities: local density, local energy and the associated components of the current. Multiplying equations (1a) and (1b) by 1, v^2 and v, v^3, respectively, and integrating over v we obtain both the equations of conservation and associated equations

$$\frac{\mathrm{d}}{\mathrm{d}t}N_i^+ +\partial_x J_i=0 \qquad N_i^{\mp}(t)=\int_{-\infty}^{+\infty}f^{\mp}v^i\,\mathrm{d}v$$
$$J_i(x,t)=\int_{-\infty}^{+\infty}f^-v^{i+1}\,\mathrm{d}v \qquad i=0,2 \qquad (3a)$$

$$\partial_t J_0=\nu(\tau_1-\tau_0)J_0N_0^+ \qquad \partial_t J_2=\nu(\tau_3-\tau_0)J_2N_0^+ +3J_0N_2^+(\tau_1-\tau_3) \qquad (3b)$$

where the τ_m are the moments of the cross section $\tau_m=\int_{-\pi}^{+\pi}\sigma(\theta)(\cos\theta)^m\,\mathrm{d}\theta$, N_0^+, N_2^+ being the local density and energy, while J_i are the components of the current. The general solution of (3a) is

$$J_i(x,t)=(x_0-x)N_i^-(t)+P_i^-(t) \qquad i=0,2 \qquad (4)$$

and for simplicity we restrict ourselves here to $P_i^- \equiv 0$ (3*a*) and (3*b*) can be rewritten with t-dependent terms

$$\partial_t N_i^+ = N_i^- \qquad i = 0, 2 \tag{5a}$$

$$\begin{aligned} &\partial_t \ln \partial_t N_0^+ = \nu(\tau_1 - \tau_0) N_0^+ \\ &\partial_{t^2}^2 N_2^+ = \nu[(\tau_3 - \tau_0) N_0^+ \partial_t N_2^+ + 3 N_2^+ \partial_t N_0^+ (\tau_1 - \tau_3)]. \end{aligned} \tag{5b}$$

Equations (5*a*) and (5*b*) form a system of closed equations for N_i^+, N_i^- or J_i. We remark that if $f(2\pi)^{1/2} \to \exp(-v^2/2)$ (Maxwellian equilibrium) then from their definitions, $N_i^+ \to 1$ when $t \to \infty$.

There exists for N_0^+ (5*b*) a general solution of exponential time type:

$$N_0^+ = \frac{1-y}{1+y} \xrightarrow[t\to\infty]{} 1 \qquad y = c_1 \exp(-c_2 t) \qquad \frac{c_2}{\nu(\tau_0 - \tau_1)} = 1 \tag{6a}$$

c_1, $c_2 > 0$ being arbitrary constants; and a particular one of power type

$$N_0^+ = \frac{2}{\nu(\tau_0 - \tau_1)} \frac{1}{t + t_0} \xrightarrow[t\to\infty]{} 0 \tag{6b}$$

which when substituted into (5*b*) for N_2^+ and into (5*a*) for N_i^- lead to the complete determination of both N_i^+, J_i in the semi-inhomogeneous Kac model. For the local density associated to a Maxwellian equilibrium (6*a*), if $c_1 > 0$ ($c_1 < 0$), $N_0^+ \to 1^\mp$ when $t \to \infty$, $\mathrm{d}N_0^+/\mathrm{d}t = 2c_2 y(1+y)^{-2}$, we have contraction (dilatation). For $N_0^+ \to 0$ given by (6*b*) the associated stationary solution is necessarily $f \equiv 0$. Now we study the corresponding N_2^+,

(i) $N_0^+ = (1-y)/(1+y)$: we have $J_0 = (x - x_0) 2c_2 y(1+y)^{-2}$ and for the local energy a second-order differential equation in the y variable

$$\left(\partial_{y^2}^2 + \frac{[2y - \mu(1-y)]}{y(1+y)} \partial_y - \frac{6\mu}{y(1+y)^2}\right) N_2^+ = 0 \qquad y = c_1 \exp(-c_2 t) \qquad \mu = \frac{\tau_1 - \tau_3}{\tau_0 - \tau_1} \tag{7a}$$

and due to $\sigma(\theta) > 0$, then $-\frac{1}{4} < \mu < 2$. Equation (7*a*) has two independent solutions $N_{21}^+ = 1 + \Sigma_1^\infty \tilde{a}_n y^n$, $N_{22}^+ = y^{1+\mu}(1 + \Sigma_1^\infty \tilde{\tilde{a}}_n y^n)$, analytic for $|y| < 1$, the sets $(\tilde{a}_n)$, $(\tilde{\tilde{a}}_n)$ satisfying a three-term recursion relation. The general solution $N_2^+ \to 1$ when $t \to \infty$ is a linear combination $N_2^+ = N_{21}^+ + c_3 N_{22}^+$. If $\mu = 1$, the general solution contains, as usual, a lg y term. If $\mu = 0$ or $\tau_1 = \tau_3$, the solution becomes $N_2^+ = \mathrm{const}_1 + \mathrm{const}_2(1+y)^{-1}$. The only general constraint is $N_2^+ > 0$ leading to available c_1, c_2, c_3 interval values. Once N_2^+ is determined we deduce the second current component $J_2 = (x_0 - x)\partial_t N_2^+$.

(ii) $N_0^+ = 2[\nu(\tau_0 - \tau_1)(t + t_0)]^{-1}$ leads for the general N_2^+ to a linear combination of two power-type solutions

$$\begin{aligned} &N_2^+ = c_3 (t + t_0)^{\rho_+} + c_4 (t + t_0)^{\rho_-} \\ &2\rho_\pm = -(1 + 2\mu) \pm (1 + 4\mu^2 - 20\mu)^{1/2} \qquad \mu = \frac{\tau_1 - \tau_3}{\tau_0 - \tau_1}. \end{aligned} \tag{7b}$$

$\sigma(\theta) > 0$, the reality of $\rho_\pm$ and the fact that $N_2^+ \to 0$ when $t \to \infty$ (the only possible stationary solution being $f \equiv 0$) lead to the result $\rho_\pm < 0$ if $0 < \mu < \frac{5}{2} - \sqrt{6}$. We notice that in both cases (i) and (ii) the only microscopic constraint on $\sigma(\theta)$ enters into μ defined in equation (7*a*) and (7*b*).

Another interesting macroscopic quantity is the average flow velocity $U_0(x,t)=J_0(x,t)/N_0^+=(x_0-x)\partial_t \ln N_0^+(t)$ and we find for the above two cases:

(i) $f \xrightarrow[t\to\infty]{}$ a Maxwellian $\qquad U_0=(x_0-x)2C_2y/(1-y^2) \underset{t\to\infty}{\simeq} O(\exp(-C_2 t))$ (8*a*)

(ii) $f \xrightarrow[t\to\infty]{} 0 \qquad U_0=-(x_0-x)/(t+t_0) \underset{t\to\infty}{\simeq} O(t^{-1}).$ (8*b*)

This is all that we can predict at the macroscopic level. It remains to show, at the microscopic level, that there exist $\sigma(\theta)$ interactions and solutions $f(v,x,t)$ which satisfy these macroscopic conditions. Simple odd-velocity distributions f^- fulfilling the current relations (4) are provided with:

$$f^-(v,x,t)=(x_0-x)f^-(v,t)+g^-(v,t) \tag{9}$$

and for simplicity we restrict our study to $g^-=0$.

3. Distributions relaxing to zero when $t\to\infty$

In this section we construct a class of solutions $f^\mp(v,t)$ with inverse time dependence for N_i^+, J_i. Since we are not interested in the more general solution, for simplicity we assume that the $f^\pm$ are degenerate, being the product of v-dependent functions $F^\pm(v)$ by functions of t. Our aim is to find, for a finite interval, let us say $|x_0-x|<a$, that there exist both $f(v,t,x)>0\ \forall t, v$ and $\sigma(\theta)>0$. At the beginning we do not assume any special symmetry for the cross section, which means that the odd moments of $\sigma(\theta)$ are different from zero ($\tau_{2p+1}\neq 0$).

From the analysis of § 2, it follows that at the macroscopic level we have: $N_0^+=2[\nu(\tau_0-\tau_1)(t_0+t_1)]^{-1}$, $J_0=2(x-x_0)[\nu(\tau_0-\tau_1)(t_0+t_1)^2]^{-1}$, $N_2^+=N_0^+d_2^+/d_0^+$, $J_2=J_0d_2^-/d_0^-$ with $d_i^+=\int_{-\infty}^{+\infty}F^+v^i\,dv$, $d_i^-=\int_{-\infty}^{\infty}F^-v^{1+i}\,dv$, $i=0,2$; $d_0^+d_2^-=d_0^-d_2^+$ and at the microscopic level we have for the distribution:

$$f(v,t,x)=\frac{2}{(\tau_0-\tau_1)(t+t_0)}\left(\frac{F^+(v)}{d_0^+}+\frac{(x-x_0)F^-(v)}{(t+t_0)d_0^-}\right) \qquad t_0>0. \tag{10a}$$

Since N_0^+ is decreasing, f corresponds to an *expansion* case. However we must look at the positivity property of f. From (10*a*), we show that if $f>0$ at $t=0$ and x fixed, then $f>0$ for all values of t. Firstly we remark that necessarily $F^+>0$ for all v values, otherwise $F^-(v)$ for either positive or negative v values being negative, the bracket on the RHS of (10*a*) will become negative. Secondly, in this bracket, the term proportional to F^- is strictly decreasing and its absolute value is maximum at $t=0$. On the other hand, if for v, t fixed and $|F^-(v)|\neq 0$, we allow $|x-x_0|$ to become arbitrarily large, the bracket will become negative. In conclusion, if there exists an interval $x\in[x_0-a, x_0+a]$, $a>0$ with $f>0$ at $t=0$, then the positivity property of f is preserved for all $t>0$ values. In this section we restrict x to stay inside such intervals and in § 5 discuss for x outside these intervals.

Substituting f given by (10*a*) into the Kac equations (1*a*), (1*b*), leads to the $\sigma(\theta)$, v-dependent integral relations between F^+/d_0^+ and F^-/d_0^- where t has disappeared.

$$[-1+2\tau_0/(\tau_0-\tau_1)]\frac{F^+(v)}{d_0^+}+\frac{vF^-(v)}{(d_0^-)}=\frac{2}{(\tau_0-\tau_1)(d_0^+)^2}\iint \sigma(\theta)F^+(v')F^+(w')\,dw\,d\theta \tag{10b}$$

$$d_0^+ \tau_1 F^-(v) = \iint \sigma(\theta) F^-(v') F^+(w') \, \mathrm{d}w \, \mathrm{d}\theta. \tag{10c}$$

We do not discuss the general solution $F^\pm$ of (10*a*), (10*b*) and (10*c*) fulfilling the positivity constraint on f. For simplicity we further assume the symmetry $\sigma(\theta) = \sigma(\pi - \theta)$ or $\tau_m = 0$, for m odd. Then (10*c*) becomes identically zero on both sides and we are left with only (10*b*) that we substitute into (10*a*). Finally f can be written in closed form as a functional of an arbitrary function $F^+(v)$:

$$f(v, t, x) = \frac{2}{(t_0 + t)\tau_0} \left[\frac{F^+(v)}{d_0^+} + \frac{(x - x_0)}{v (d_0^+)^2 (t + t_0)} \times \left(\frac{2}{\tau_0} \iint \sigma(\theta) F^+(v') F^+(w') \, \mathrm{d}w \, \mathrm{d}\theta - F^+(v) d_0^+ \right) \right] \tag{11}$$

the second term of the bracket on the RHS of (11) being $(x - x_0) F^-(v)/d_0^-$. We have two constraints on F^+: first it must be such that $F^- \to 0$ when $v \to 0$ and second $|F^-/F^+| \xrightarrow[v \to \infty]{} 0$ (or const). The first condition leads to

$$2\tau_0^{-1} \int_{-\pi}^{+\pi} \int_{-\infty}^{+\infty} F^+(w \sin\theta) F^+(w \cos\theta) \sigma(\theta) \, \mathrm{d}w \, \mathrm{d}\theta - F^+(0) d_+ = 0.$$

For the second we remark that if $F^+ \simeq_{v \to \infty} \exp(-b_0 v^2) v^{-2\alpha_+}$ then $\int F^+(v') F^+(w') \sigma(\theta) \, \mathrm{d}\theta \, \mathrm{d}w \simeq \exp(-b_0 v^2) v^{-4\alpha_+}$ and it follows that F^+ dominates at large v if $\alpha_+ > -\frac{1}{2}$. For instance very simple families are given by: $F^+(v) = \exp(-bv^2)(\Sigma_0^{n_+} a_n v^{2n})^{-\alpha}$, $a_n > 0$, $n_+ \alpha > -\frac{1}{2}$. Combining these two constraints we can obtain solutions $f > 0$ in (11). For example in figure 1(*a*) we plot the $vf(v, t, x)$ relaxation curves for

$$F^+(v) = \exp(-b_0 v^2)(1 + v^2)^{-\alpha} \tag{11'}$$

with $b_0 = 0.4121 \times 10^{-3}$, $\alpha = \frac{1}{2}$, $t_0 = 1$, $\tau_0 = 1$ and $\sigma(\theta) = \frac{1}{4} \Sigma_1^2 [\delta(\theta - \theta_i) + \delta(\theta + \theta_i)]$, $\theta_1 = \pi/4$, $\theta_2 = 3\pi/4$ and $x - x_0 = 1$. In conclusion there exist distributions $f(v, t, x) > 0$ for $|x - x_0|$ finite and such that $f \to 0$ when $t \to \infty$. The width of the interval $|x - x_0| < a$ is such that $f > 0 \; \forall t$, v depends on the model. For instance in figure 1(*b*), for the same example as in figure 1(*a*), we present in the x, v plane, the domain where f violates positivity for $t = 0, 1, \ldots$ and we verify that the maximal domain is obtained for $t = 0$.

4. Distributions relaxing towards a Maxwellian

An exact solution has recently been obtained (Cornille 1984b). In order to have a self-contained paper we start here, in the semi-inhomogeneous formalism, with slightly different initial assumptions and briefly present the arguments leading to its determination. Assuming that $f^\pm$ are products of a Gaussian $\exp(-v^2/2\Delta(t))$ by v polynomials, then from equation (1), they necessarily are $\alpha_0(t) + \alpha_2(t) v^2/2$ for f^+ and $\alpha_1(t) v/\sqrt{2}$ for f^-. Instead of α_0, α_1, α_2, we can use the macroscopic quantities: $\alpha_0 = 1.5 N_0^+ \Delta^{-1/2} - 0.5 N_2^+ \Delta^{-3/2}$, $\alpha_2 = N_2^+ \Delta^{-5/2} - N_0^+ \Delta^{-3/2}$, $\alpha_1 = 2\sqrt{2}\, \Delta^{-3/2} \partial_t N_0^+$. It remains to determine Δ. From the definition (3*a*) of the current components J_n it follows that $J_2 = 3 J_0 \Delta$. Substituting this relation into equation (3) we find N_2^+ and $\partial_t N_2^+$ in terms of Δ, N_0^+:

$$N_2^+ = N_0^+ \Delta + [\nu(\tau_1 - \tau_3)]^{-1} \partial_t \ln \Delta \qquad \partial_t N_2^+ = 3 \Delta \partial_t N_0^+.$$

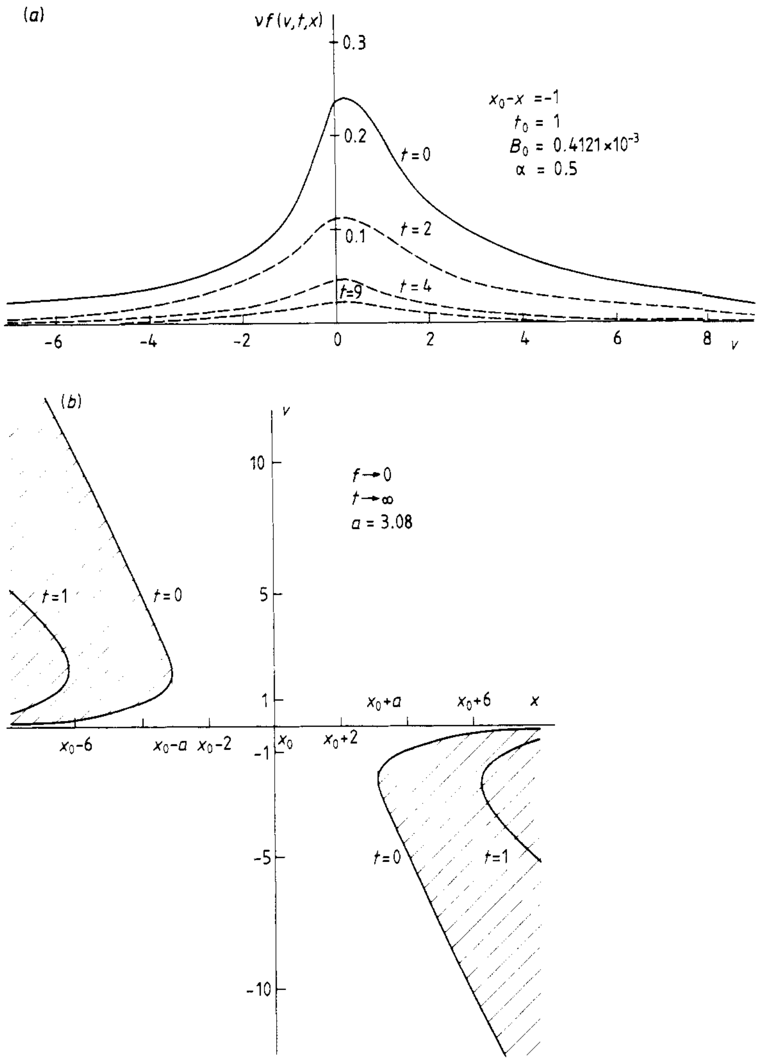

Figure 1. (a) Plot of $\nu f(v, t, x)$ against v given in equations (11) and (11′). (b) Positivity domains in the x, v plane for different t values of $\nu f(v, t, x)$ given by equations (11) and (11′).

We eliminate N_2^+ and obtain a second-order differential equation for Δ, entirely determined when N_0^+ is known:

$$\{2\partial_t N_0^+ - N_0^+\partial_t - [\nu(\tau_1 - \tau_3)]^{-1}\partial_{t^2}^2\}\Delta(t) = 0.$$

These results are general and do not depend on any particular choice of the local density. For instance for f having a Maxwellian relaxation or $N_0^+ = (1-y)/(1+y)$, we obtain for Δ:

$$\left[\frac{4\mu}{y(1+y)^2} + \left(\frac{\mu(1-y)-(1+y)}{y(1+y)}\right)\partial_y - \partial_{y^2}^2\right]\Delta(y) = 0 \qquad \mu = \frac{\tau_1 - \tau_3}{\tau_0 - \tau_1}. \tag{12a}$$

Let us define $\sigma_2=\tau_2-\tau_4$, then in (1*a*) the coefficients of the v^4 terms give: $N_2^+\Delta^{-5/2}-N_0^+\Delta^{-3/2}=\alpha_2=(\Delta^2\nu\sigma_2)^{-1}(\partial_t\ln\Delta)$ and a second relation $N_2^+=N_0^+\Delta+(\sigma_2\nu)^{-1}\partial_t\ln\Delta$. It follows that the cross section must satisfy a moments relation

$$\sigma_2-\tau_1+\tau_3=0 \qquad \mu=\sigma_2(\tau_0-\tau_1)^{-1}\in\,]0,2[\tag{12b}$$

and from Δ, N_0^+, N_2^+ we determine the $\alpha_i(t)$ and f. Choosing (6*a*) for N_0^+, we get:

$$(2\pi)^{1/2}f(v,t,x)=\exp(-\tfrac{1}{2}v^2\Delta^{-1})\Delta^{-3/2}\left[\left(\frac{1-y}{1+y}\right)\Delta+\frac{y}{2\mu}\partial y\Delta\right.$$
$$\left.+2v\frac{c_2(x_0-x)y}{(1+y)^2}-\frac{v^2}{2}y\partial y\ln\Delta\right]. \tag{12c}$$

(Equations similar to (12*a*)-(12*c*) for N_0^+ in (6*b*), violate positivity either for f or $\sigma(\theta)$). Choosing $\Delta(y)\to1$ when $y\to0^+$ in (12*a*), then $(2\pi)^{1/2}f$ tends to the Maxwellian $\exp(-v^2/2)$ when $t\to\infty$. As in (7*a*), the general Δ solution is a linear combination of two solutions $\Delta=(1+\Sigma_1^\infty a_ny^n)+c_3y^\mu(1+\Sigma_1^\infty d_ny^n)$, $\mu\neq1$, with the summations valid for $|y|<1$, c_3 being a constant, the sets $(a_n(\mu),(d_n(\mu))$ satisfy a three-term recursion relation and can be numerically computed. Here we restrict ourselves to $C_3=0$ and for the study of the Tjon overpopulation effect of high-velocity particles at intermediate times (Tjon 1979) we define the reduced distribution $F(v,t,x)=f(v,t,x)/f(v,\infty,x)\to1$ when $t\to\infty$. For $|v|$, t large, $F-1$ is approximated by a positive v, t function multiplied by sgn $(1-\mu)$. Then independently of the initial conditions, the effect can occur only if the microscopic $\sigma(\theta)$ model satisfies $\mu<1$ and this is numerically observed. In figure 2(*a*) we plot the relaxation curves for $\mu=0.25$, $c_1=-0.1$, $c_2=1$, $c_3=0$, $x_0-x=1$ and $\sigma(\theta)=\Sigma_1^2\mu_i[\delta(\theta-\theta_i)+\delta(\theta+\theta_i)]$, $\cos\theta_1=-\cos\theta_2=\sqrt{\mu}$, $\mu_2=\mu_1(1-\sqrt{\mu})(1+\sqrt{\mu})^{-1}$. In figure 2(*b*), for the same example as in figure 2(*a*) we draw in the x, v plane the domains where f violates positivity for $t=0, 1, 1.5, \ldots$ and we still verify that the maximal domain is obtained at $t=0$. For $C_3=0$, new interesting features appear.

Let us discuss the properties of the exact solution (12*a*), (12*b*), (12*c*) which can be classified following the μ values (or the $\sigma(\theta)$ models).

(i) $\mu>1$. The positivity of f at $t=0$ requires $C_1>0$ (or $y>0$). Consequently N_0^+ increases (see § 2) corresponding to a *contraction* case. The standard continuity argument can be modified (Cornille 1984b) proving that positivity at $t=0$ means positivity at $t>0$. Consequently in the x, v plane, the domain violating positivity is maximal at $t=0$ and there always exists $|x-x_0|<a$ such that $f>0\;\forall t, v$. The Tjon effect does not exist.

(ii) $\mu<1$. The positivity at $t=0$ requires $C_1<0$ (or $y<0$); N_0^+ decreases and consequently (§ 2) we are in an *expansion* case. However if the solution has two relaxation times, or $C_3\neq0$, the positivity for all t values requires $C_3<0$ (Cornille 1984b). Consequently for $C_3\leqslant0$ (see for instance the example of figure 2) there exists an interval $|x_0-x|<a$ with $f>0$ for all t, v values. The Tjon effect exists only if $C_3=0$.

(iii) $\mu=1$. This corresponds to a transition where the fundamental solutions of the Fuchsian equation (12*a*) have a $\ln y$ term. The positivity of f for all t values requires $C_1>0$ (or $y>0$), it corresponds to a *contraction* case and there exists $|x_0-x|<a$ with $f>0\;\forall t, v$. The Tjon effect does not exist.

Finally we notice that the exact solution (12*a*), (12*b*), (12*c*) can also be obtained from the differential Laguerre moments systems (Cornille 1984b), then giving up the search for closed solutions one could try to obtain other solutions of the differential systems not too far from the present one.

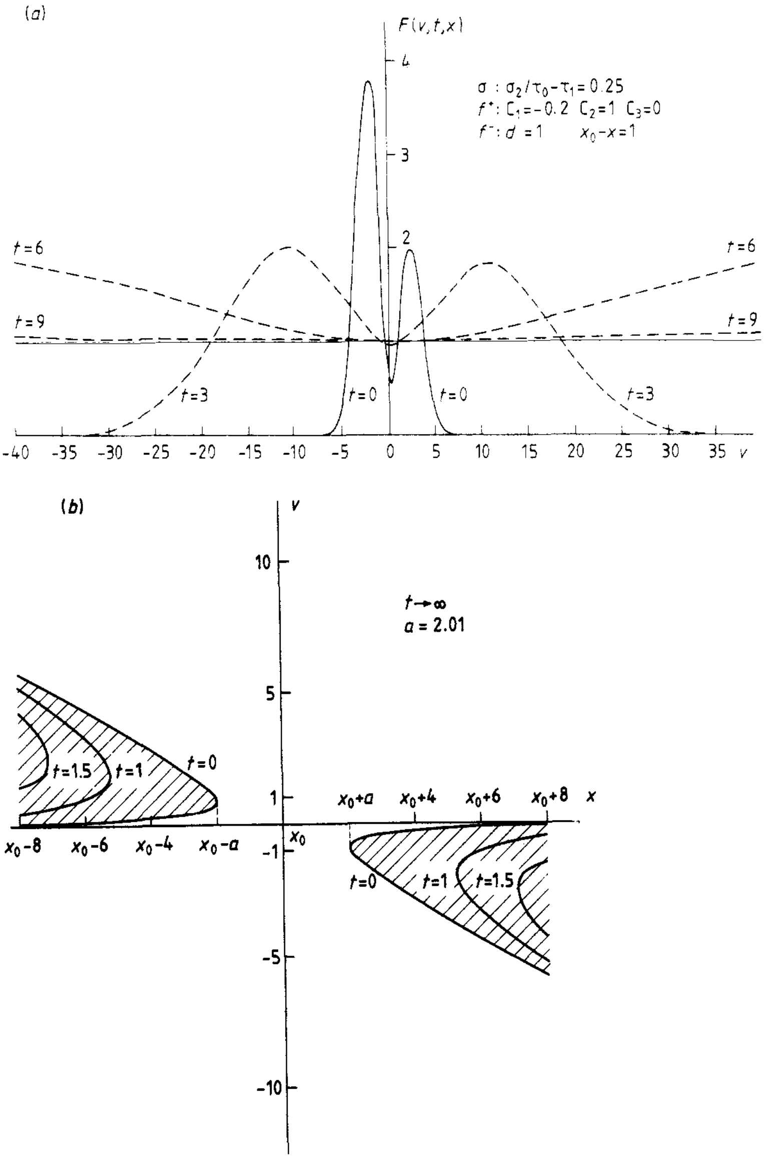

Figure 2. (*a*) Plot of $f(v, t, x)$ against v, given in equations (12). (*b*) Positivity domains in the x, v plane for different t values of $f(v, t, x)$ given by equations (12). $(2\pi)^{1/2}f \rightarrow \exp(-v^2/2)$.

5. A physical interpretation of the solutions $f(v, x, t)$

We start with the distributions of §§ 3 and 4. $f = f(v, x, t) = f^+(v, t) + (x_0 - x)f^-(v, t)$ which satisfy the Kac equation: $(\partial_t + v\partial_x)f = \mathrm{col}(f)$. We define a new distribution $\tilde{f} = f\theta[a^2 - (x_0 - x)^2] \geq 0$ for $|x_0 - x|^2 < a^2$ *and identically zero for z outside* $(x_0 - a, x_0 + a)$. We find $(\partial_t + v\partial_x)\tilde{f} = \mathrm{col}(\tilde{f}) + v(S^+ + S^-)$ and obtain for the collision term two supplementary ones that we want to interpret:

$$vS^+ = vf^+(v, t)\{-\delta[x - (x_0 + a)] + \delta[x - (x_0 - a)]\} \tag{13}$$

$$vS^- = vf^-(v, t)a\{\delta[x-(x_0+a)]+\delta[x-(x_0-a)]\}. \tag{14}$$

In both cases, with Maxwellian relaxation or without, $f^+(v, t)$ dominates over $f^-(v, t)$ for large t, whence the same result for vS^+ compared with vS^-.

5.1. vS^+

We recall that necessarily $f^+(v, t) \geq 0$ (otherwise adding f^- will lead to a violation of the positivity constraint on f), and vf^+ has the sign of v. For $x = x_0 + a$ (or $x = x_0 - a$), we have a *sink* for $v > 0$ (or a source) and a *source* (or a sink) for $v < 0$. The amount of incoming and outgoing particles being the same, vS^+ can be viewed *as elastic walls* at $x = x_0 \pm a$. For large t we find:

(i) f given by equations (12) and relaxing to a Maxwellian

$$vS^+ \underset{t\to\infty}{\simeq} v\exp(-v^2/2)(-\delta[x-(x_0+a)]+\delta[x-(x_0-a)] \tag{13a}$$

the elastic walls are always present.

(ii) f given by (10a) and going to zero when $t \to \infty$

$$vS^+ = \frac{2F^+(v)}{d_0^+(\tau_0-\tau_1)(t_0+t)}\{-\delta[x-(x_0+a)]+\delta[x-(x_0-a)]\} \underset{t\to\infty}{\simeq} O(t^{-1}) \tag{13b}$$

therefore vS^+ becomes smaller and smaller when t increases.

5.2. vS^-

A priori for $v > 0$ (or $v < 0$) vf^- does not have a well defined sign. However for equations (12) it is positive in the contraction case and negative in the expansion one.

(i) In the Maxwellian relaxation equations (12)

$$vS^- = C_1\left(\frac{av^2\exp(-v^2/2\Delta)2C_2\exp(-C_2t)}{\Delta^{3/2}(1+y)^2}\right)\{\delta[x-(x_0+a)]+\delta[x-(x_0-a)]\} \tag{14a}$$

the first bracket on the RHS is positive and vS^- has the sign of C_1. If $C_1 > 0$, *contraction* case (or $C_1 < 0$, *expansion* case) then vS^- is a *source* (or a *sink*) at both $x = x_0 \pm a$, which disappears for large t:

$$vS^- \underset{t\to\infty}{\simeq} \exp(-C_2t)[2C_2C_1v^2\exp(-v^2/2)]\{\delta[x-(x_0+a)]+\delta[x-(x_0-a)]\}.$$

(ii) In the non-Maxwellian relaxation case (10a)

$$vS^- = \frac{2vF^-(v)}{(\tau_0-\tau_1)(t_0+t)^2d_0^-}\{\delta[x-(x_0+a)]+\delta[x-(x_0-a)]\} \underset{t\to\infty}{\simeq} O(t^{-2}) \tag{14b}$$

the situation is different because $vF^-(v)$ for either $v > 0$ or $v < 0$ can change sign (for instance this happens for the example of figure 1(a)). Consequently it is difficult to interpret this term physically. On the other hand it is never negligible compared with the elastic walls.

In conclusion vS^-, the source or sink term, is negligible compared with the elastic walls ($S^-/S^+ = O(\exp(-C_2t))$) in the case of relaxation with Maxwellian equilibrium. On the contrary the physical interpretation of vS^- is unclear in the non-Maxwellian relaxation and it always gives an important contribution ($S^-/S^+ = O(t^{-1})$). Similarly the average flow velocity (see equations (8)) decreases exponentially in the Maxwellian relaxation case and only like t^{-1} in the non-Maxwellian one. We can therefore expect that in a more realistic problem, with ordinary walls, there exist solutions looking like those with Maxwellian relaxation rather than the others.

At the end we notice that the main difference between our solutions and the Nikolskii ones (besides the existence of solutions with Maxwellian relaxation) is that they really depend on *three variables*. On the contrary, inhomogeneous solutions deduced from homogeneous ones still depend on *two variables*. Can this semi-inhomogeneous formalism be extended to more realistic models is an open problem.

Another interesting aspect of the Boltzmann equation is in the connection with the non-integrable equations. In the Maxwellian interaction formalism, equivalent two-dimensional nonlinear partial differential equations exist (Krook and Wu 1976) and the simple exact solutions appear as solitons and bisolitons. A whole class of nonlinear non-integrable equations exist (Cornille 1983 and references therein) with common properties with the BE. Here we find *exact three-dimensional solutions* and it seems interesting to find and study the nonlinear equation associated with the Kac model.

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